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Second-Order Linearity of the General Signed-Rank Statistic

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Let $X_1,..., X_n$ be i.i.d. random variables symmetric about zero. Let $R_i(t)$ be the rank of $|X_i - tn^{-1/2}|$ among $|X_1 - tn^{-1/2}|,..., |X_n - tn^{-1/2}|$ and $T_n(t) = \sum_{i=1}^{n} \varphi((n+1)^{-1} R_i(t)) \operatorname{sign}(X_i - tn^{-1/2})$. We show that there exists a sequence of random variables V_n such that $\sup_{0 \leqslant t \leqslant 1} |T_n(t) - T_n(0) - tV_n| \to 0$ in probability, as $n \to \infty$. V_n is asymptotically normal. © 1987 Academic Press, Inc.

I. INTRODUCTION AND RESULT

Some functional limit theorems in mathematical statistics (cf. [1, 3, 7, 9]) assert the weak convergence of a sequence $Z_n(t)$, $0 \leqslant t \leqslant 1$, of stochastic processes to a limiting Gaussian process $Z(t)$ different from usual limiting processes. The trajectories of $Z(t)$ do not display the highly irregular behavior of Brownian paths, but are, on the contrary, highly regular. We shall consider a situation in which $Z(t) = t \cdot Y$, with a normally distributed random variable Y. These theorems are usually proven by means of the well-developed machinery of weak convergence in function spaces, but there are other possibilities. Our approach consists essentially in showing that $\sup_{0 \leqslant t \leqslant 1} |Z_n(t) - tZ_n(1)| \to 0$ in probability and then establishing the weak convergence of $Z_n(1)$.

We describe now the subject of this paper. Let $X_1,..., X_n$ be n independent copies of a random variable X distributed symmetrically about 0. Let t be a real number. We denote by $R_i(t)$ the rank of $|X_i - tn^{-1/2}|$ among $|X_1 - tn^{-1/2}|,..., |X_n - tn^{-1/2}|$. Consider the signed-rank statistic

$$T_n(t) = \sum_{i=1}^{n} \varphi\left(\frac{R_i(t)}{n+1}\right) \operatorname{sign}(X_i - tn^{-1/2}), \tag{1}$$

where φ: $[0, 1) \to \mathbb{R}$ is a given score-function. The asymptotic properties of T_n are well known: Under suitable conditions $n^{-1/2}T_n(0)$ is asymptotically

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normally distributed, and $n^{-1/2}(T_n(t)-T_n(0)) \to -ct$ in probability, as $n \to \infty$, for some $c>0$. Sometimes it is desirable to have preciser result on $T_n(t)$. Van Eeden [12], in analogy with result of Jurečková [8] on linear rank statistics, was able to prove that $\sup_{0\leqslant t\leqslant 1} |n^{-1/2}(T_n(t)-T_n(0))+ct| \to 0$ in probability. In the Wilcoxon-case, $\varphi \equiv \mathrm{id}$, going one step further, Antille [1] showed that the stochastic process $Z_n(t)=(T_n(t)-T_n(0))+cn^{1/2}t$, $0\leqslant t\leqslant 1$, converges to a stochastic process of the form $t\cdot Y$, as described above. (Jurečková derived the corresponding result for linear rank statistics [9].) We shall derive the corresponding result for general score functions φ. A similar result was obtained independently and by different methods by Hušková und Jurečková [6] and, for the case of linear rank statistics, by Hušková [4]. There are several applications of these assertions. For example, Antille [1, 2] uses his results to study the Hodges–Lehmann estimator. We do not go further into this and refer the reader to the cited papers.

We formulate now the main assumptions.

(A) *The score function* φ: $[0,1) \to \mathbb{R}$ *is three times continuously differentiable and* $\varphi(0)=0$.

Let $F(x)=\Pr(X_i\leqslant x)$ denote the distribution function of X_i, $F^{-1}(x)$ its inverse.

(B) $F(x)$ *possesses a density* $f(x)$, *symmetric about zero and twice differentiable with bounded second derivative. The set of all* x *with* $f(x)>0$ *is an interval, possibly the whole real line. Finally, letting*

$$g(t)=f\left(F^{-1}\left(\frac{t+1}{2}\right)\right), \qquad -1<t<1, \tag{2}$$

we have

$$g'(t)=O((1-t)^{-1}g(t)) \qquad \text{as} \quad t\to 1.$$

(C′) *There are numbers* λ, ρ *such that*

(i) $\varphi^{(i)}(t)=O((1-t)^{-\lambda-i})$ as $t\to 1$, $i=0,1,2,3$. (3)

($\varphi^{(i)}$ *denotes the ith derivative of* φ, $\varphi^{(0)}\equiv\varphi$.)

(ii) $g(t)=O((1-t)^{\rho})$ *as* $t\to 1$, (4)

(iii) $\lambda>0$, $\rho>\frac{3}{2}\lambda+\frac{1}{2}$. (5)

This assumption may be replaced by

(C″) $\varphi^{(i)}(t)$ *is bounded on* $[0,1)$ *for* $i=0,1,2,3$.

Now let

$$k(t)=4g(t)\,\varphi'(t)-2\int_0^t u\,g(u)\,\varphi''(u)\,du-2\int_t^1 (u-1)\,g(u)\,\varphi''(u)\,du. \tag{6}$$

In view of (C') $k(t)$ is well defined. Further let

$$U_i = 2F(|X_i|) - 1. \tag{7}$$

U_i is uniformly distributed on $[0, 1]$.

THEOREM. *Let* (A), (B) *and either* (C′) *or* (C″) *be satisfied. Then for any* $M > 0$,

$$\sup_{|t| \leq M} \left| T_n(t) - T_n(0) + tn^{-1/2} \sum_{i=1}^{n} k(U_i) - 2tn^{1/2} \int_0^1 \varphi'(u)\, g(u)\, du \right| \to 0$$

in probability, as $n \to \infty$.

The proof is given in Section 2. Note that in view of (C′) $k(t) = O((1-t)^{-1/2+\lambda/2})$, as $t \to 1$. Therefore $k(U_i)$ has finite second moments. A functional limit theorem for the process $T_n(t) - T_n(0)$ follows immediately.

Let us examine the assumptions of the theorem a little more closely. If $\varphi(0) \neq 0$, we may apply the theorem using the score function $\varphi(t) - \varphi(0)$ instead. The remainder term $\varphi(0) \sum_i \{\mathrm{sign}(X_i - tn^{-1/2}) - \mathrm{sign}(X_i)\}$, is of order $n^{1/4}$, so the theorem is no longer true. The same happens if $\varphi(t)$ possesses discontinuities, as was first noted and analyzed by Hušková and Jurečková [5]. Concerning (B), note that, from the assumption on the support of f, $g(t) > 0$ for $-1 < t < 1$, but $\lim_{t \to 1} g(t) = 0$. Thus the assumption on $g'(t)$ means that $g(t)$ is not too irregular close to 1. The main assumptions of the theorem are thus contained in (C′).

Here is an example. Let us suppose that $f(x) > 0$ everywhere. Then F^{-1} is unbounded on $(0, 1)$, and its derivative $1/g(2t-1)$ is not finitely integrable on $(0, 1)$. Thus, if $g(t)$ is not too irregular (as in the case of the normal density), $g(t) = O((1-t)^{\rho})$ for any $\rho < 1$, and any $\lambda < \frac{1}{3}$ is permitted in (3). This includes van der Waerden's score function.

Finally, we comment on the problem of ties. $R_i(t)$ is well defined with probability one for each fixed t, but with positive probability there are values of t such that ties occur among the $|X_k - tn^{-1/2}|$, making it undefined. There are various proposals on how to define the ranks in this situation, but they have no effect on the validity of our theorem, as will be shown in the remark in the Appendix. To be definite we define $R_i(t)$ so that $T_n(t)$ has right-continuous paths.

II. PROOF OF THE THEOREM

The proof divides into two major parts. In the first one, using a Taylor expansion, a truncated version of $T_n(t) - T_n(0)$ is shown to be asymptotically linear in the sense of the theorem. In the second part we show that truncation is permitted. Since the proof is rather involved, it may be helpful for the reader to consider the Wilcoxon case, $\varphi \equiv \mathrm{id}$, first. Then Lemmas 1, 2, and 5 are sufficient for the proof.

We assume throughout the proof that φ together with its first three derivatives are non-negative, since in general there are functions φ_1, φ_2, possessing these properties additionally to our requirements, such that $\varphi = \varphi_1 - \varphi_2$. (Put $\varphi_1''' = \max(0, \varphi''')$, $\varphi_2''' = \max(0, -\varphi''')$, $\varphi_1''(0) = \max(0, \varphi''(0))$ and so forth.) From (B)

$$|\Pr(2sn^{-1/2} \leqslant x + X_1 \leqslant 2tn^{-1/2}) - 2(t-s)\, n^{-1/2} f(x)| \leqslant C(t-s)\, n^{-1} \quad (8)$$

for some $C > 0$ as well as

$$\left| \Pr(2sn^{-1/2} \leqslant X_1 + X_2 \leqslant 2tn^{-1/2}) - 2(t-s)\, n^{-1/2} \int f^2 \right| \leqslant C(t-s)\, n^{-1} \quad (9)$$

since $X_1 + X_2$ has the density $\int f(x-y) f(y)\, dy$.

LEMMA 1. *Let $U_n(t)$, $V_n(t)$, $0 \leqslant t \leqslant 1$, $n = 1, 2, \ldots$, be two sequences of stochastic processes with increasing path and $U_n(0) = V_n(0) = 0$ a.s. Let $Z_n(t) = U_n(t) - V_n(t)$. If there exist real numbers γ_n such that $\gamma_n = O(n^{-k})$ for all $k > 0$, if there are positive numbers β, δ, C and random variables $V_n \geqslant 0$ such that for $0 \leqslant s < t \leqslant 1$,*

$$\operatorname{var}(Z_n(t) - Z_n(s)) \leqslant Cn^{-\delta}(t-s) + \gamma_n,$$
$$|EU_n(t) - EU_n(s)| \leqslant Cn^{\beta}(t-s),$$
$$|V_n(t) - V_n(s)| \leqslant (t-s)\, V_n \qquad \text{a.s.},$$
$$EV_n = O(n^{\beta}),$$

then in probability, as $n \to \infty$,

$$\sup_{0 \leqslant t \leqslant 1} |Z_n(t) - EZ_n(t)| \to 0.$$

Proof. Fix $\lambda > 0$, let $N = N_n$ be the largest natural number smaller than n^{λ} and let $t_v = vN^{-1}$. Since $U_n(t)$ and $V_n(t)$ have increasing paths

$$\begin{aligned}
&\sup_{0 \leqslant t \leqslant 1} (Z_n(t) - EZ_n(t)) \\
&\quad \leqslant \max_{1 \leqslant v \leqslant N} [U_n(t_v) - V_n(t_{v-1}) - EU_n(t_{v-1}) + EV_n(t_v)] \\
&\quad \leqslant \max |Z_n(t_v) - EZ_n(t_v)| + N^{-1}(V_n + Cn^{\beta}).
\end{aligned}$$

If $\lambda > \beta$, the second term goes to zero. Further, if n is large enough, $\operatorname{var}(Z(t_v) - Z(t_\mu)) \leqslant 2C(t_v - t_\mu)\, n^{-\delta}$ for $0 \leqslant \mu < v \leqslant N$. From Lemma II.7 of Antille [1]

$$E\{\max_v |Z_n(t_v) - EZ_n(t_v)|^2\} = O(n^{-\delta}(\log n)^2).$$

A similar estimate of $\sup_t (EZ_n(t) - Z_n(t))$ completes the proof. ∎

Now, let

$$R_i = R_i(0),$$

$$C_i(t) = \operatorname{card}\{j \neq i \mid 0 < X_i + X_j \leqslant 2tn^{-1/2}\} = \sum_{j \neq i} I(0 < X_i + X_j \leqslant 2tn^{-1/2}),$$

where $I(A)$ denotes the indicator function of the event A. It is easy to establish for $k = 1,\dots, 4$,

$$EC_i(1)^k = O(n^{k/2}). \tag{10}$$

LEMMA 2. *For suitable* $0 < s_i < 1$, *we have*

$$\begin{aligned} T_n(t) - T_n(0) = & -(n+1)^{-1} \sum_i \varphi'\left(\frac{R_i}{n+1}\right) C_i(t) \\ & + \frac{1}{2}(n+1)^{-2} \sum_i \varphi''\left(\frac{R_i}{n+1}\right) C_i(t)^2 \operatorname{sign} X_i \\ & - \frac{1}{6}(n+1)^{-3} \sum_i \varphi'''(s_i)\, C_i(t)^3 + r_n(t), \end{aligned}$$

where in probability, as $n \to \infty$, $\sup_{0 \leqslant t \leqslant 1} |r_n(t)| \to 0$.

Proof. In the case $\varphi \equiv \mathrm{id}$ the assertion follows from formula (3.16), p. 129, in Lehmann [10], with $r_n(t) = 0$ a.s. Thus we may replace $\varphi(t)$ by $\varphi(t) - t\varphi'(0)$, in other words, we may assume $\varphi'(0) = 0$. If now $X_i > n^{-1/2}$, $0 \leqslant t \leqslant 1$,

$$\begin{aligned} R_i(t) &= \operatorname{card}\{1 \leqslant j \leqslant n \mid |X_j - tn^{-1/2}| < |X_i - tn^{-1/2}|\} + 1 \\ &= \operatorname{card}\{j \mid 2tn^{-1/2} < X_i + X_j < 2X_i\} + 1, \end{aligned}$$

thus

$$R_i(t) - R_i(0) = -C_i(t) \operatorname{sign} X_i.$$

We derive the same formula also in the case $X_i < -n^{-1/2}$, if we proceed from

$$R_i(t) = \mathrm{card}\{j \mid |X_j - tn^{-1/2}| \leqslant |X_i - tn^{-1/2}|\}$$
$$= \mathrm{card}\{j \mid 2X_i \leqslant X_i + X_j \leqslant 2tn^{-1/2}\}.$$

(Only in the case of ties both expressions differ. They are chosen such that $T_n(t)$ has rightcontinuous paths.) Now, by means of a Taylor expansion, using the notation $\sum^* a_i = \sum a_i I(|X_i| > n^{-1/2})$,

$$\begin{aligned}
&\sum_i{}^* \varphi\left(\frac{R_i(t)}{n+1}\right) \mathrm{sign}(X_i - tn^{-1/2}) - \sum_i{}^* \varphi\left(\frac{R_i}{n+1}\right) \mathrm{sign}\, X_i \\
&= -(n+1)^{-1} \sum_i{}^* \varphi'\left(\frac{R_i}{n+1}\right) C_i(t) \\
&\quad + \frac{1}{2}(n+1)^{-2} \sum_i{}^* \varphi''\left(\frac{R_i}{n+1}\right) C_i(t)^2\, \mathrm{sign}\, X_i \\
&\quad - \frac{1}{6}(n+1)^{-3} \sum_i{}^* \varphi'''(s_i)\, C_i(t)^3
\end{aligned} \tag{11}$$

with suitable $0 < s_i < 1$. It remains to remove the *. To this end let A_n be the event that for at least $cn^{1/2}$ of all $i \in \{1, ..., n\}$ $|X_i| \leqslant 3n^{-1/2}$ holds. $\Pr(A_n) = o(1)$, if c is large enough. On the complementary event of $A_n |X_i| \leqslant n^{-1/2}$ entails $R_i(t) \leqslant cn^{1/2}$ for $0 \leqslant t \leqslant 1$. Since $\varphi'(0) = 0$, for a suitable $d > 0$, on A_n^c

$$\sup_t \left| (n+1)^{-1} \sum_i I(|X_i| \leqslant n^{-1/2})\, \varphi'\left(\frac{R_i}{n+1}\right) C_i(t) \right|$$
$$\leqslant dn^{-3/2} \sum_i I(|X_i| \leqslant n^{-1/2})\, C_i(1) = o_p(1)$$

in view of (10). This shows that we may remove the * at the first term of the right-hand side of (11). The other terms are treated similarly. The left-hand side of (11) is handled by the observation that on A_n^c, since $\varphi(0) = \varphi'(0) = 0$,

$$\sup_t \left| \sum_i \varphi\left(\frac{R_i(t)}{n+1}\right) \mathrm{sign}(X_i - tn^{-1/2})\, I(|X_i| \leqslant n^{-1/2}) \right|$$
$$\leqslant \sum \varphi(cn^{-1/2})\, I(|X_i| \leqslant n^{-1/2}) = o_p(1). \quad \blacksquare$$

Next, we introduce the function φ_n: $[0, 1) \to \mathbb{R}$ by defining its third derivative as

$$\varphi_n'''(t) = \min\{n^\alpha, \varphi'''(t)\},$$

where $0 < \alpha < \frac{1}{8}$, and then integrating it three times such that $\varphi_n^{(i)}(0) = \varphi^{(i)}(0)$ for $i = 0, 1, 2$. Note $0 \leqslant \varphi_n, \varphi_n', \varphi_n'', \varphi_n''' \leqslant n^\alpha$.

LEMMA 3. *In probability, for any* $0 < s_i < 1$,

$$n^{-2} \sup_{0 \leqslant t \leqslant 1} \left| \sum_i \varphi_n''\left(\frac{R_i}{n+1}\right) C_i^2(t) \operatorname{sign} X_i \right| \to 0,$$

$$n^{-3} \sup_{0 \leqslant t \leqslant 1} \left| \sum_i \varphi_n'''(s_i)\, C_i^3(t) \right| \to 0.$$

Proof. The second assertion is immediate from (10) and

$$\sup_{0 \leqslant t \leqslant 1} \left| \sum_i \varphi_n'''(s_i)\, C_i^3(t) \right| \leqslant n^\alpha \sum_i C_i^3(1) = O_p(n^{\alpha + 5/2}).$$

Let us consider

$$Z_{n1}(t) = n^{-2} \sum_i \varphi_n''(U_i)\, C_i^2(t)\, I(X_i > 0) = n^{-2} \sum_i \sum_{k, l \neq i} V_{ikl}(t),$$

where

$$V_{ikl}(t) = \varphi_n''(U_i)\, I(0 < X_i + X_k \leqslant 2tn^{-1/2}, 0 < X_i + X_l \leqslant 2tn^{-1/2}, X_i > 0),$$

$$V_{ikl}(s, t) = V_{ikl}(t) - V_{ikl}(s).$$

It is always assumed that $i \neq k, l$. In order to apply Lemma 1 to $Z_{n1}(t)$, we have to estimate $\operatorname{var}(Z_{n1}(t) - Z_{n1}(s))$. If i, k, l, j, p, q are different from each other, $\operatorname{cov}(V_{ikl}(s, t), V_{jpq}(s, t)) = 0$ by independence. It is not diffucult to convince oneself that in case of $\operatorname{card}\{i, k, l, j, p, q\} = 5$

$$\operatorname{cov}(V_{ikl}(s, t), V_{jpq}(s, t)) \leqslant D|t - s|\, n^{2\alpha - 2}$$

and in any case

$$\operatorname{var}(V_{ikl}(s, t)) \leqslant D|t - s|\, n^{2\alpha - 1/2},$$

with some $D > 0$. Therefore

$$\begin{aligned} \operatorname{var}(Z_{n1}(t) - Z_{n1}(s)) &\leqslant n^{-4} \sum_{i \neq k, l} \sum_{j \neq p, q} \operatorname{cov}(V_{ikl}(s, t), V_{jpq}(s, t)) \\ &\leqslant n^{-4}(Dn^5|t - s|\, n^{-2 + 2\alpha} + Dn^4|t - s|\, n^{2\alpha - 1/2}). \end{aligned}$$

Using (9),

$$|EZ_{n1}(t) - EZ_{n1}(s)| \leqslant C|t-s|\, n^{\alpha+1/2}.$$

From Lemma 1 in probability

$$\sup_{0 \leqslant t \leqslant 1} |Z_{n1}(t) - EZ_{n1}(t)| \to 0.$$

Now using (8),

$$\begin{aligned} &E(\varphi_n''(U_1)\, C_1(t)^2 I(X_1 > 0)) \\ &\quad = n(n-1) \int_0^\infty \varphi_n''(2F(|x|)-1) \Pr\{0 < x + X_2 \leqslant 2tn^{-1/2}\}^2 f(x)\, dx \\ &\quad = n\, a_n\, t^2 + O(n^{\alpha+1/2}) \end{aligned}$$

with $a_n = 4 \int_0^\infty \varphi_n''(2F(|x|)-1) f^3(x)\, dx$, uniformly in $0 \leqslant t \leqslant 1$. Thus $\sup_t |EZ_{n1}(t) - a_n t^2| \to 0$, and it follows

$$\sup_{0 \leqslant t \leqslant 1} \left| n^{-2} \sum_j \varphi_n''(U_i)\, C_i^2(t)\, I(X_i > 0) - a_n t^2 \right| \to 0$$

in probability. The result remains valid, if $I(X_i > 0)$ is replaced by $I(X_i \leqslant 0)$, therefore

$$\sup_{0 \leqslant t \leqslant 1} \left| n^{-2} \sum_j \varphi_n''(U_i)\, C_i^2(t) \operatorname{sign} X_i \right| \to 0.$$

Finally, using (10) and Lemma 8,

$$\begin{aligned} &\sup_t \left| n^{-2} \sum_i \left(\varphi_n''(U_i) - \varphi_n''\left(\frac{R_i}{n+1}\right)\right) C_i^2(t) \operatorname{sign} X_i \right| \\ &\quad \leqslant n^{-2} \sup_t \varphi_n'''(t) \sum_i \left| U_i - \frac{R_i}{n+1} \right| C_i^2(1) = O_p(n^{\alpha-1/2}), \end{aligned}$$

and the desired result follows. ∎

LEMMA 4. *In probability*

$$n^{-1} \sup_{0 \leqslant t \leqslant 1} \left| \sum_i \left(\varphi_n'\left(\frac{R_i}{n+1}\right) - \varphi_n'(U_i) \right) (C_i(t) - 2tn^{1/2} f(X_i)) \right| \to 0.$$

Proof. Let

$$Q_i = I\left(\varphi_n'\left(\frac{R_i}{n+1}\right) \geqslant \varphi_n'(U_i)\right),$$

$$\begin{aligned} Z_{n2}(t) &= n^{-1} \sum_i \left(\varphi_n'\left(\frac{R_i}{n+1}\right) - \varphi_n'(U_i)\right) C_i(t)\, Q_i \\ &\quad - n^{-1} \sum_i \left(\varphi_n'\left(\frac{R_i}{n+1}\right) - \varphi_n'(U_i)\right) 2tn^{1/2} f(X_i)\, Q_i \\ &= U_{n2}(t) - V_{n2}(t) \qquad \text{(say)}. \end{aligned}$$

Q_i is introduced to ensure increasing paths of $U_{n2}(t)$ and $V_{n2}(t)$. Now,

$$\begin{aligned} &E((Z_{n2}(t) - Z_{n2}(s))^2) \\ &\quad \leqslant E\left\{\left(\varphi_n'\left(\frac{R_1}{n+1}\right) - \varphi_n'(U_1)\right)^2 (C_1(t) - C_1(s) - 2(t-s)\, n^{1/2} f(X_1))^2\right\} \\ &\quad \leqslant n^{2+2\alpha} \Pr\left(\left|\frac{R_1}{n+1} - U_1\right| \geqslant n^{-3/8}\right) \\ &\qquad + \sup_t \varphi_n''(t)^2\, n^{-3/4} E(C_1(t) - C_1(s) - 2(t-s)\, n^{1/2} f(X_1))^2 \\ &\quad \leqslant \gamma_n + c|t-s|\, n^{2\alpha - 1/4}. \end{aligned}$$

$\gamma_n = O(n^{-k})$ for any $k > 0$ by Lemma 8 (Appendix). Further, by means of (9),

$$EU_{n2}(t) - EU_{n2}(s) \leqslant n^{\alpha-1} \sum_i E\{C_i(t) - C_i(s)\} \leqslant Cn^{\alpha+1/2}|t-s|.$$

Thus from Lemma 1 and

$$|EZ_{n2}(t)|^2 \leqslant EZ_{n2}^2(t) \leqslant \gamma_n + cn^{2\alpha - 1/4} = o(1)$$

we get $\sup_t |Z_{n2}(t)| \to 0$ in probability. The same conclusion is valid, if Q_i is replaced by $I(\varphi_n'(R_i/(n+1))) < \varphi_n'(U_i))$. Combining both results, the assertion follows. ∎

LEMMA 5. *In probability*

$$\sup_{0 \leqslant t \leqslant 1} \left| n^{-1} \sum_i \varphi_n'(U_i)\, C_i(t) - 4tn^{-1/2} \sum_i \varphi'(U_i)\, g(U_i) \right.$$
$$\left. + 2tn^{1/2}\{2E\varphi'(U_1)\, g(U_1) - E\varphi_n'(U_1)\, g(U_1)\} \right| \to 0.$$

Proof. Denote

$$Z_{n3}(t) = n^{-1} \sum_i \varphi_n'(U_i)(C_i(t) - 2tC_i(\tfrac{1}{2})).$$

Fix $s < t$ and let

$$\bar{Z}_{ij} = I(2sn^{-1/2} < X_i + X_j \leqslant 2tn^{-1/2}) - 2(t-s)\, I(0 < X_i + X_j \leqslant n^{-1/2}).$$

From (8),

$$E(\bar{Z}_{12}\bar{Z}_{13}) \leqslant C'(t-s)^2 n^{-2},$$

thus with a suitable $D > 0$,

$$\begin{aligned}
\operatorname{var}(Z_{n3}(t) - Z_{n3}(s)) &= \operatorname{var}\left(n^{-1} \sum_i \sum_{i \neq j} \varphi_n'(U_i)\, \bar{Z}_{ij}\right) \\
&\leqslant E(\varphi_n'(U_1)^2 \bar{Z}_{12}^2) + 4nE(\varphi_n'(U_1)^2 \bar{Z}_{12}\bar{Z}_{13}) \\
&\leqslant Dn^{2\alpha - 1/2}(t-s).
\end{aligned}$$

Next we show $EZ_{n3}(t) \to 0$ uniformly in t. To this end

$$\begin{aligned}
&\int_{-\infty}^{\infty} \varphi_n'(2F(|x|) - 1)\{2tn^{-1/2} f(x) - \Pr(0 \leqslant x + X_1 \leqslant 2tn^{-1/2})\}\, f(x)\, dx \\
&= \int_{-\infty}^{\infty} \varphi_n'(\ldots)\, f(x) \int_0^{2tn^{-1/2}} \left(f'(-x)\, y + f''(\cdots)\frac{y^2}{2}\right) dy\, dx \\
&= O(n^{\alpha - 3/2}), \qquad (12)
\end{aligned}$$

since $\int \varphi_n'(2F(|x|) - 1)\, f(x)\, f'(x)\, dx = 0$. Thus uniformly in t

$$\begin{aligned}
|EZ_{n3}(t)| = n \bigg| \int_{-\infty}^{\infty} & \varphi_n'(2F(|x|) - 1)\{\Pr(0 \leqslant x + X_1 \leqslant 2tn^{-1/2}) \\
& - 2t \Pr(0 \leqslant x + X_1 \leqslant n^{-1/2})\}\, f(x)\, dx \bigg| = O(n^{\alpha - 1/2}).
\end{aligned}$$

From Lemma 1 we get

$$\sup_t |Z_{n3}(t)| \to 0 \qquad (13)$$

in probability. Next we consider

$$n^{-1} \sum_i \varphi_n'(U_i)\, C_i(\tfrac{1}{2}) = \tfrac{1}{2} n^{-1} \sum_{i \neq j} (\varphi_n'(U_i) + \varphi_n'(U_j))\, I(0 < X_i + X_j \leqslant n^{-1/2}).$$

We introduce the function

$$G_n(x)=\varphi_n'(2F(|x|)-1)\Pr(0\leqslant x+X_1\leqslant n^{-1/2})$$
$$+E\{\varphi_n'(U_1)\,I(0\leqslant x+X_1\leqslant n^{-1/2})\}$$

and the random variables

$$U_{ij}=(\varphi_n'(U_i)+\varphi_n'(U_j))\,I(0<X_i+X_j\leqslant n^{-1/2})-G_n(X_i)-G_n(X_j)+EG_n(X_1).$$

Then $E(U_{ij}\mid X_i)=E(U_{ij}\mid X_j)=0$ and $\operatorname{cov}(U_{ij},U_{kl})=0$ in case of $\{i,j\}\neq\{k,l\}$, consequently $\operatorname{var}(n^{-1}\sum_{i\neq j}U_{ij})\leqslant EU_{12}^2=O(n^{2\alpha-1/2})$, thus

$$n^{-1}\sum_i\varphi_n'(U_i)\,C_i\left(\frac{1}{2}\right)=\frac{n-1}{n}\sum_i G_n(X_i)-\frac{(n-1)}{2}EG_n(X_1)+o_p(1). \quad (14)$$

Using (8), it is not difficult to show that, uniformly in x,

$$G_n(x)-2\varphi_n'(2F(|x|)-1)\,n^{-1/2}f(x)=O(n^{\alpha-1}),$$

further from (12)

$$\begin{aligned}EG_n(X_1)&-2E\{\varphi_n'(U_1)\,g(X_1)\}\,n^{-1/2}\\&=2\int_{-\infty}^{\infty}\varphi_n'(2F(|x|)-1)\{\Pr(0\leqslant x+X_1\leqslant n^{-1/2})\\&\quad-n^{-1/2}f(x)\}\,f(x)\,dx\\&=o(n^{-1}),\end{aligned}$$

thus from (13) and (14),

$$\begin{aligned}&n^{-1}\sum_i\varphi_n'(U_i)\,C_i(t)\\&\quad=4tn^{-1/2}\sum_i\varphi_n'(U_i)\,g(U_i)-2tn^{1/2}E\{\varphi_n'(U_1)\,g(U_1)\}+o_p(1),\end{aligned}$$

where $o_p(1)$ goes to zero in probability uniformly in t. Now the assertion of the lemma follows, if we can show $\operatorname{var}(n^{-1/2}\sum_i(\varphi_n'(U_i)-\varphi'(U_i))\,g(U_i))\to 0$. From the construction of φ_n, $0\leqslant\varphi_n'(t)\leqslant\varphi'(t)$ and $\varphi_n'(t)=\varphi'(t)$, unless $t\geqslant 1-n^{-d}$ for some $d>0$. $g(U_1)\cdot\varphi'(U_1)$ has finite second moment, therefore

$$\begin{aligned}&\operatorname{var}\left(n^{-1/2}\sum_i(\varphi_n'(U_i)-\varphi'(U_i))\,g(U_i)\right)\\&\quad\leqslant E\{(\varphi_n'(U_1)-\varphi'(U_1))^2\,g(U_1)^2\}\\&\quad\leqslant E\{\varphi'(U_1)^2\,g(U_1)^2I(U_1\geqslant 1-n^{-d})\}=o(1).\quad\blacksquare\end{aligned}$$

LEMMA 6. *In probability*

$$n^{-1/2}\sum_i \left(\varphi_n'\left(\frac{R_i}{n+1}\right)-\varphi_n'(U_i)\right) f(X_i)$$
$$=n^{-1/2}\sum_i \int_0^1 (I(U_i<y)-y)\,\varphi''(y)\,g(y)\,dy+o_p(1).$$

Proof. Using a Taylor expansion

$$n^{-1/2}\sum_i \left(\varphi_n'\left(\frac{R_i}{n+1}\right)-\varphi_n'(U_i)\right) g(U_i)$$
$$=n^{-1/2}\sum_i \left(\frac{R_i}{n+1}-U_i\right)\varphi_n''(U_i)\,g(U_i)$$
$$+\frac{1}{2}n^{-1/2}\sum_i \left(\frac{R_i}{n+1}-U_i\right)^2 \varphi_n'''(\bar{U}_i)\,g(U_i)$$
$$=S_{n1}+S_{n2} \qquad \text{(say)}$$

with $\bar{U}_i\in(0,1)$. Since $\max_{1\leqslant i\leqslant n}|R_i/(n+1)-U_i|^2=O_p(n^{-1})$, in probability $S_{n2}=O_p(n^{\alpha-1/2})=o_p(1)$. On the other hand S_{n1} is asymptotically equivalent to

$$S_{n1}^*=n^{-1}\sum_{i\neq j} h_n(U_i,U_j),$$

where

$$h_n(U_i,U_j)=(I(U_i>U_j)-U_i)\,\varphi_n''(U_i)\,g(U_i)$$

has the properties

$$E(h_n(U_i,U_j)\mid U_i)=0,$$
$$E(h_n(U_i,U_j)\mid U_j)=\int_0^1 (I(u>U_j)-u)\,\varphi_n''(u)\,g(u)\,du,$$
$$E(h_n(U_i,U_j)-E(h_n(U_i,U_j)\mid U_j))^2\leqslant Eh_n^2(U_i,U_j)\leqslant dn^{2\alpha}$$

for some $d>0$. Therefore

$$E\left(S_{n1}^*-n^{-3/2}\sum_{i\neq j}E(h_n(U_i,U_j)\mid U_j)\right)^2=O(n^{2\alpha-1}),$$

which shows the asymptotic equivalence of S_{n1} and $n^{-1/2}\sum_i\int_0^1(I(U_i<u)-u)\,\varphi_n''(u)\,g(u)\,du$. We leave it to the reader to prove that, using (3)–(5), φ_n'' may be replaced by φ''. ∎

Combining Lemma 2–6, we are now in the position to prove our theorem under assumption (C″), since in this case $\varphi_n \equiv \varphi$ for large n. The following lemma enables us to treat the case of unbounded score functions, too.

LEMMA 7. *Let $\psi_n(t) = \varphi(t) - \varphi_n(t)$. Then in probability*

$$\sup_{0 \leqslant t \leqslant 1} \left| \sum_i \psi_n\left(\frac{R_i(t)}{n+1}\right) \operatorname{sign}(X_i - tn^{-1/2}) - \sum_i \psi_n\left(\frac{R_i}{n+1}\right) \operatorname{sign} X_i \right.$$
$$\left. + 2tn^{1/2} E\{\psi_n'(U_1)\, g(U_1)\} \right| \to 0.$$

Proof. From the construction of φ_n, for some $0 < \mu < 1$,

$$\psi_n(t) = 0, \qquad \text{if} \quad (1-t) \geqslant n^{\mu - 1}. \tag{15}$$

Therefore $\Pr(\operatorname{sign}(X_i - tn^{-1/2}) \neq \operatorname{sign} X_i$ and $\psi_n(R_i(t)/(n+1)) \neq 0$ for some t and $i) \to 0$ as $n \to \infty$, and it suffices to show

$$\sup_{0 \leqslant t \leqslant 1} \left| \sum_i \left(\psi_n\left(\frac{R_i(t)}{n+1}\right) - \psi_n\left(\frac{R_i}{n+1}\right)\right) \operatorname{sign} X_i + 2tn^{1/2} \int \psi_n' g \right| \to 0. \tag{16}$$

We rearrange the order of summation. Define V_i and $D_i(t)$ by

$$V_{R_i} = \operatorname{sign} X_i, \qquad D_{R_i}(t) = C_i(t).$$

V_i are i.i.d. random variables with values 0 or 1, each occuring with probability $\frac{1}{2}$, and independent of $|X_1|, \ldots, |X_n|$ and thus $R_1, \ldots, R_n$. Since $X_i = |X_i| \operatorname{sign} X_i$

$$D_i(t) = \operatorname{card}\{ j \neq i \mid 0 < |X|_{(i)} V_i + |X|_{(j)} V_j \leqslant 2tn^{-1/2} \},$$

where $|X|_{(1)} \leqslant \cdots \leqslant |X|_{(n)}$ are the order statistics of $|X_1|, \ldots, |X_n|$. Now fix $d > 0$ and suppose $|X|_{(i)} \geqslant d$. Then, if n is large enough, $0 < |X|_{(i)} V_i + |X|_{(j)} V_j \leqslant 2n^{-1/2}$ can only occur, if $V_i \neq V_j$. Letting

$$\eta_i = I(V_i = -1), \qquad \bar{\eta}_i = I(V_i = 1),$$
$$\zeta_{in}(t) = \operatorname{card}\{ j \mid i < j \leqslant n,\ |X|_{(j)} \leqslant |X|_{(i)} + 2tn^{-1/2},\ \eta_j = 0\},$$
$$\bar{\zeta}_{in}(t) = \operatorname{card}\{ j \mid 1 \leqslant j < i,\ |X|_{(j)} \geqslant |X|_{(i)} - 2tn^{-1/2},\ \bar{\eta}_j = 0\},$$

we see that $|X|_{(i)} \geqslant d$ entails $D_i(t) = \eta_i \zeta_{in}(t) + \bar{\eta}_i \bar{\zeta}_{in}(t)$. Let

$$M_n = \left\{ \psi_n\left(\frac{R_i(t)}{n+1}\right) \neq 0 \text{ and } |X_i| \leqslant d \text{ for some } 1 \leqslant i \leqslant n,\ 0 \leqslant t \leqslant 1 \right\}.$$

It is not difficult to check in view of (15) that $\Pr(M_n) \to 0$, if d is small enough. On the complement of M_n we have, using $R_i(t) = R_i - C_i(t) \operatorname{sign} X_i$,

$$\begin{aligned}
&\sum_i \left\{ \psi_n\left(\frac{R_i(t)}{n+1}\right) - \psi_n\left(\frac{R_i}{n+1}\right)\right\} \operatorname{sign} X_i \\
&\quad = \sum_i \left\{ \psi_n\left(\frac{i}{n+1} - \frac{1}{n+1} V_i D_i(t)\right) - \psi_n\left(\frac{i}{n+1}\right)\right\} V_i \\
&\quad = -(n+1)^{-1} \sum_i \psi_n'\left(\frac{i}{n+1}\right)(\eta_i \zeta_{in}(t) + \bar{\eta}_i \bar{\zeta}_{in}(t)) \\
&\qquad + \frac{1}{2}(n+1)^{-2} \sum_i \psi_n''(\tau_{in})(\eta_i \zeta_{in}(t) + \bar{\eta}_i \bar{\zeta}_{in}(t))^2 V_i,
\end{aligned} \tag{17}$$

with $|\tau_{in} - (i/(n+1))| \leqslant n^{-1} D_i(t) \leqslant n^{-1}(\zeta_{in}(t) + \bar{\zeta}_{in}(t))$.

Now, let $0 < T_1 < T_2 < \cdots$ be the jump times of a standard Poisson process and $G(x) = F^{-1}((1+x)/2)$. Since $G(x)$ is the inverse distribution function of $|X_1|$, $|X|_{(1)}, \ldots, |X|_{(n)}$ and $G(T_1 T_{n+1}^{-1}), \ldots, G(T_n T_{n+1}^{-1})$ have the same common distributions. Therefore we may apply Lemmas 11 and 12 to the terms on the right-hand side of (17), and (16) follows. ∎

III. Appendix

The following result is well known; the proof might be new.

Lemma 8. *Let $U_1, \ldots, U_n$ be independent and uniformly distributed on $[0, 1]$, R_1 the rank of U_1 among $U_1, \ldots, U_n$. For any natural number k there is a $C_k > 0$ such that $E(U_1 - (R_1/(n+1)))^{2k} \leqslant C_k n^{-k}$.*

Proof. There is a $D_k > 0$ such that for any $x \in (0, 1)$ $E(\sum_{i=1}^n I(U_i \leqslant x) - nx)^{2k} \leqslant D_k n^k$. Since $R_1 - 1 = \sum_{j=2}^n I(U_j < U_1)$,

$$E\left(U_1 - \frac{R_1 - 1}{n-1}\right)^{2k} = E(E((\cdots)^{2k} \mid U_1)) \leqslant D_k (n-1)^{-k},$$

from which the result follows. ∎

The following lemmas deal with the jump times $0 \equiv T_0 < T_1 < T_2 < \cdots$ of a standard Poisson process, in other words, $T_{i+1} - T_i$ are independent, exponentially distributed random variables. By $\eta_1, \eta_2, \ldots,$ we denote a sequence of i.i.d. random variables, which is independent of the sequence $T_1, T_2, \ldots$, and for which $\Pr(\eta_i = 0) = \Pr(\eta_i = 1) = \frac{1}{2}$. Finally let

$$\rho_{in}(t) = \text{card}\left\{ j \mid i < j \leqslant n,\ \eta_j = 0,\ T_j \leqslant T_i + 2tn^{1/2}\, g\left(\frac{i}{n+1}\right) \xi_{in}(t) \right\}$$

with random $\xi_{\text{in}}(t) > 0$.

LEMMA 9. *Suppose that for any $\delta > 0$ and any $0 < \nu < \mu < 1$ with $\mu/2 < \nu$ in probability*

$$\sup\{ |\xi_{\text{in}}(t) - 1| : n - n^{\mu} \leqslant i \leqslant n - n^{\nu},\ 0 \leqslant t \leqslant 1 \} = O_p(n^{\mu/2 - \nu + \delta}). \tag{18}$$

Then there exists a $\varepsilon > 0$ such that for any ν, μ with $(\rho - \frac{1}{2})(\rho + 1)^{-1} - \varepsilon < \nu < \mu < 1$,

$$\sup_{0 \leqslant t \leqslant 1} \left| n^{-1} \sum_{i = n - n^{\mu}}^{n - n^{\nu}} \varphi'\left(\frac{i}{n+1}\right) \eta_i \rho_{in}(t) - \tfrac{1}{2} t n^{1/2} \int_{1 - n^{\mu - 1}}^{1 - n^{\nu - 1}} \varphi' g \right| \to 0.$$

Proof. We start with some preliminary formulas. Let

$$N_k(t) = \text{card}\{ j > k \mid T_j \leqslant T_k + t,\ \eta_j = 0 \}, \qquad N_k(s, t) = N_k(t) - N_k(s).$$

(For convenience, $N_k(t) = 0$, if $t < 0$.) $N_k(t)$ is a Poisson process with intensity rate $\frac{1}{2}$, independent of $T_1, \ldots, T_k$. One easily verifies that for $a, b > 0$,

$$4\, \text{cov}(N_k(a), N_k(b - T_k)) = E \min(a, (b - T_k)^+),$$

with $a^+ = \max(0, a)$. Now let $0 < a < b$, $0 < c < d$, $k \geqslant 1$. Using $N_0(a, b) = \sum_{i=1}^{k} I(a < T_i < b) + N_k(a - T_k, b - T_k)$,

$$\begin{aligned}
&\text{cov}\left(\eta_i \left(N_i(a, b) - \frac{b-a}{2} \right), \eta_{i+k} \left(N_{i+k}(c, d) - \frac{d-c}{2} \right) \right) \\
&= \text{cov}\left(\eta_0 N_0(a, b), \eta_k N_k(c, d) - \eta_k \frac{d-c}{2} \right) \\
&\quad = \text{cov}\left(\eta_0 N_k(a - T_k, b - T_k), \eta_k N_k(c, d) - \eta_k \frac{d-c}{2} \right) \\
&\quad = E(\eta_0 \eta_k)\, \text{cov}(N_k(a - T_k, b - T_k), N_k(c, d)) \\
&\quad = \tfrac{1}{16} E\{ \min(d, (b - T_k)^+) - \min(c, (b - T_k)^+) \} \\
&\qquad - \tfrac{1}{16} E\{ \min(d, (a - T_k)^+) - \min(c, (a - T_k)^+) \} \\
&\quad \leqslant (d - c) \Pr(T_k \leqslant b).
\end{aligned} \tag{19}$$

Further

$$\operatorname{var}\left(\eta_i\left(N_i(a,b)-\frac{b-a}{2}\right)\right)\leqslant \operatorname{var}(N_i(a,b))\leqslant (b-a)\Pr(T_0\leqslant b). \quad (20)$$

We use these formulas to study

$$Z_{n4}(t)=n^{-1}\sum_i{}'\,\varphi'\left(\frac{i}{n+1}\right)\eta_i N_i\left(2tn^{1/2}g\left(\frac{i}{n+1}\right)\right)$$
$$-tn^{-1/2}\sum_i{}'\,\varphi'\left(\frac{i}{n+1}\right)\eta_i\, g\left(\frac{i}{n+1}\right).$$

The dash indicates that the sum is taken over all i with $n-n^{\mu}\leqslant i\leqslant n-n^{\nu}$. We show

$$\sup_{0\leqslant t\leqslant 1}|Z_{n4}(t)|\to 0, \quad (21)$$

in probability, if $(\rho-\frac{1}{2})(\rho+1)^{-1}-\varepsilon<\nu<\mu<1$, where $\varepsilon>0$ is determined later. It is no restriction to require in the sequel additionally $\mu/2<\nu$, $\mu\leqslant b_1$ for some $b_1<1$ and $\mu-\nu\leqslant b_2$, where $b_2>0$ will be chosen in dependence of b_1. By means of (3), (4), (19), (20), and the formula $\sum_{j=0}^{\infty}\Pr(T_j\leqslant c)=1+c$, if $0\leqslant s\leqslant t\leqslant 1$, we get for a suitable $D>0$,

$$\begin{aligned}
&\operatorname{var}(Z_{n4}(t)-Z_{n4}(s))\\
&\quad\leqslant 2n^{-2}\sum_i{}'\sum_{j\geqslant i}{}'\,\varphi'\left(\frac{i}{n+1}\right)\varphi'\left(\frac{j}{n+1}\right)\\
&\qquad\times 2(t-s)\,n^{1/2}g\left(\frac{j}{n+1}\right)\Pr\left(T_{j-i}\leqslant 2n^{1/2}g\left(\frac{i}{n+1}\right)\right)\\
&\quad\leqslant Dn^{-3/2}(t-s)\,n^{-(\nu-1)(2+2\lambda)}n^{\rho(\mu-1)}\sum_i{}'\sum_{j\geqslant i}\Pr\left(T_{j-i}\leqslant 2n^{1/2}g\left(\frac{i}{n+1}\right)\right)\\
&\quad\leqslant D(t-s)(n^{p}+Dn^{q})
\end{aligned}$$

with $p=-\frac{1}{2}+(\mu-1)(\rho-1-2\lambda)+(\mu-\nu)(2+2\lambda)$ and $q=(\mu-1)(2\rho-1-2\lambda)+(\mu-\nu)(2+2\lambda)$. From (5) $q\leqslant(b_1-1)\,\lambda+b_2(2+2\lambda)<0$, if b_2 is small enough. Also $p<0$, if $\rho-1-2\lambda\geqslant 0$ and b_2 is small. Thus let $\rho-1-2\lambda<0$. From (5), if $\varepsilon>0$ is small enough, $-\frac{1}{2}+((\rho-\frac{1}{2})/(\rho+1)-\varepsilon-1)(\rho-1-2\lambda)<0$, thus $-\frac{1}{2}+(\mu-1)(\rho-1-2\lambda)<0$, if $\mu\geqslant(p-\frac{1}{2})/(\rho+1)-\varepsilon$, and $p<0$ follows, if again b_2 is small enough. $Z_{n4}(t)$ thus fulfills the variance condition of Lemma 1. The other conditions are easily checked, therefore (21) follows, since $EZ_{n4}(t)=0$. Now

$$\operatorname{var}\left(n^{-1/2}\sum_i{}' \varphi'\left(\frac{i}{n+1}\right)\eta_i g\left(\frac{i}{n+1}\right)\right)$$
$$=\frac{1}{4}n^{-1}\sum_i{}' \varphi'\left(\frac{i}{n+1}\right)^2 g\left(\frac{i}{n+1}\right)^2=O(n^q)$$

with $q<0$ as above, and we may replace $n^{-1/2}\ \sum'_i \varphi'(i/(n+1))\ \eta_i g(i/(n+1))$ by its mean $\frac{1}{2}n^{-1/2}\ \sum'_i \varphi'(i/(n+1))\ g(i/(n+1))$ in the definition of $Z_{n4}(t)$. Further in view of (B) and (C′), as $t\to 1$, $(\varphi' g)'(t)=O((1-t)^{-3/2})$, thus, if $n-n^{\mu}\leqslant i\leqslant n-n^{\nu}$,

$$\left|n^{-1}\varphi'\left(\frac{i}{n+1}\right)g\left(\frac{i}{n+1}\right)-\int_{(i-1)/n}^{i/n}\varphi' g\right|$$
$$=O\left(n^{-2}\left(1-\frac{i}{n}\right)^{-3/2}\right)=O(n^{-1/2-3/2\nu}),$$

consequently

$$\left|n^{-1/2}\sum_i{}' \varphi'\left(\frac{i}{n+1}\right)g\left(\frac{i}{n+1}\right)-n^{1/2}\int_{1-n^{\mu-1}}^{1-n^{\nu-1}}\varphi' g\right|$$
$$=O(n^{1/2+\mu}n^{-1/2-3/2\nu})=o(1),$$

if b_2 is small enough. Therefore (21) may be rewritten as

$$\sup_t\left|n^{-1}\sum_i{}' \varphi'\left(\frac{i}{n+1}\right)\eta_i N_i\left(2tn^{1/2}g\left(\frac{i}{n+1}\right)\right)-\frac{t}{2}n^{1/2}\int_{1-n^{\mu-1}}^{1-n^{\nu-1}}\varphi' g\right|\to 0. \tag{22}$$

Next let $\xi_n(t)$ satisfy $\sup_t|\xi_n(t)-1|=O_p(n^{(\mu/2)-\nu+\delta})=o_p(1)$ for any $\delta>0$. From (3) and (4),

$$\sup_t|\xi_n(t)-1|\,n^{1/2}\int_{1-n^{\mu-1}}^{1-n^{\nu-1}}\varphi' g=O_p(n^s),$$

where $s=(\mu-1)(\rho-\lambda-\frac{1}{2})+(\mu-\nu)(2+\lambda)+\delta<0$ in view of (5), if b_2 and δ are small enough. From (22),

$$\sup_t\left|n^{-1}\sum_i{}' \varphi'\left(\frac{i}{n+1}\right)\eta_i N_i\left(2tn^{1/2}g\left(\frac{i}{n+1}\right)\xi_n(t)\right)-\frac{t}{2}n^{1/2}\int_{1-n^{\mu-1}}^{1-n^{\nu-1}}\varphi' g\right|\to 0 \tag{23}$$

in probability. Let finally $\xi_{in}(t)$ be random variables, which obey (18). Denoting $\underline{\xi}_n(t)=\min_i \xi_{in}(t)$, $\bar{\xi}_n(t)=\max_i \xi_{in}(t)$

$$\sum_i{}' \varphi'\left(\frac{i}{n+1}\right)\eta_i N_i\left(2tn^{1/2}g\left(\frac{i}{n+1}\right)\underline{\xi}_n(t)\right)$$
$$\leqslant \sum_i{}' \varphi'\left(\frac{i}{n+1}\right)\eta_i N_i\left(2tn^{1/2}g\left(\frac{i}{n+1}\right)\xi_{in}(t)\right)$$
$$\leqslant \sum_i{}' \varphi'\left(\frac{i}{n+1}\right)\eta_i N_i\left(2tn^{1/2}g\left(\frac{i}{n+1}\right)\bar{\xi}_n(t)\right).$$

From these estimates and (23)

$$\sup_t \left| n^{-1}\sum_i{}' \varphi'\left(\frac{i}{n+1}\right)\eta_i N_i\left(2tn^{1/2}g\left(\frac{i}{n+1}\right)\xi_{in}(t)\right) - \frac{t}{2}n^{1/2}\int_{1-n^{\mu-1}}^{1-n^{\nu-1}} \varphi' g\right| \to 0.$$

In view of the definition of $N_i(t)$, to finish the proof of the lemma it remains to show that $T_n > T_i + 2tn^{1/2}\, g(i/(n+1))\ \xi_{in}(t)$ for all $i \leqslant n-n^\nu$ with probability going to 1. In view of (18), with high probability, $T_i + 2n^{1/2}\, g(i/(n+1))\ \xi_{in}(t) \leqslant T_{n-n^\nu} + cn^{1/2}n^{(\mu-1)\rho}$ for a suitable $c>0$. From the law of large numbers $n^{-\nu}(T_n - T_{n-n^\nu}) \to 1$. From (5) $\frac{1}{2}+(\mu-1)\rho<\mu$, thus $T_{n-n^\nu}+cn^{1/2}n^{(\mu-1)\rho} \leqslant T_n$ with probability going to 1, if $\mu-\nu$ is small enough. The proof is finished. ∎

LEMMA 10. *Let* $G(x)=F^{-1}((1+x)/2)$ *and* ξ_{ijn}, $1\leqslant i<j\leqslant n$, *be given by*

$$G\left(\frac{T_j}{T_{n+1}}\right)-G\left(\frac{T_i}{T_{n+1}}\right)=\frac{T_j-T_i}{2ng(i/(n+1))}\cdot\frac{1}{\xi_{ijn}}.$$

Then for any $0<\nu<\mu<1$, *such that* $\nu>\mu/2$, *and any* $\delta>0$

$$\max\left\{|\xi_{ijn}-1| : i<j\leqslant n,\ n-n^\mu\leqslant i\leqslant n-n^\nu,\ G\left(\frac{T_j}{T_{n+1}}\right) \leqslant G\left(\frac{T_i}{T_{n+1}}\right)+2n^{-1/2}\right\}=O_p(n^{\mu/2-\nu+\delta}).$$

Proof. From the mean value theorem, with $T_i/T_{n+1}\leqslant T_{ij}\leqslant T_j/T_{n+1}$,

$$G\left(\frac{T_j}{T_{n+1}}\right)-G\left(\frac{T_i}{T_{n+1}}\right)=\frac{1}{2g(T_{ij})}\cdot\frac{T_j-T_i}{T_{n+1}}. \tag{24}$$

Fix ν, μ and let $n-n^{\mu} \leqslant i \leqslant n-n^{\nu}$, $j>i$, such that

$$G\left(\frac{T_j}{T_{n+1}}\right)-G\left(\frac{T_i}{T_{n+1}}\right) \leqslant 2n^{-1/2}.$$

From (24), (4), and $T_{ij} \geqslant T_{n-n^{\mu}}/T_{n+1} \sim 1-n^{\mu-1}$,

$$\frac{T_j-T_i}{T_{n+1}} \leqslant Cn^{-1/2}\left(1-\frac{T_{n-n^{\mu}}}{T_{n+1}}\right)^{\rho} = O_p(n^{-1/2+(\mu-1)\rho}) = o_p(n^{\mu/2-1}), \tag{25}$$

since $\rho > \frac{1}{2}$ in view of (5). T_k/T_{n+1}, $k=1,..., n$, are distributed as order statistics of independent, uniformly on [0, 1] distributed random variables, therefore for any $\delta > 0$, $\max_{k \leqslant n}(1-k/n)^{1/2-\delta}|(T_k/T_{n+1})-(k/(n+1))| = O_p(n^{-1/2})$ (cf. Shorack [11, Appendix, Remark (A5)]), consequently,

$$\left|\frac{T_i}{T_{n+1}}-\frac{i}{n+1}\right| \leqslant n^{\mu/2-1+\delta}$$

for all i under consideration with probability, going to 1, as $n \to \infty$. This together with (25) entails $|T_{ij}-(i/(n+1))| \leqslant n^{\mu/2-1+\delta}$ for all i, j under consideration with probability tending to 1. Since $|1-(i/n)| \geqslant n^{\nu-1}$ and $\mu/2<\nu$, also $|1-T_{ij}| \geqslant \frac{1}{2}n^{\nu-1}$ for large n. From the mean value theorem and assumption (B), with $\bar{T}_{ij}$ between T_{ij} and $i/(n+1)$,

$$\log g(T_{ij})-\log g\left(\frac{i}{n+1}\right)=\frac{g'}{g}(\bar{T}_{ij})\left(T_{ij}-\frac{i}{n+1}\right)=O_p(n^{\mu/2-\nu+\delta}).$$

Thus $g(T_{ij})/g(i/(n+1)) = 1+O_p(n^{\mu/2-\nu+\delta})$ uniformly for all i, j under consideration, and the assertion of the lemma follows from (24) and $T_n/n = 1+O_p(n^{-1/2})$. ∎

Next denote

$$\zeta_{in}(t)=\operatorname{card}\left\{j \mid i \leqslant j \leqslant n, \eta_j=0, G\left(\frac{T_j}{T_{n+1}}\right) \leqslant G\left(\frac{T_i}{T_{n+1}}\right)+2tn^{-1/2}\right\},$$

$$\bar{\zeta}_{in}(t)=\operatorname{card}\left\{j \mid 1 \leqslant j < i, \eta_j=0, G\left(\frac{T_j}{T_{n+1}}\right) \geqslant G\left(\frac{T_i}{T_{n+1}}\right)-2tn^{-1/2}\right\}.$$

LEMMA 11. *Let $\psi_n=\varphi-\varphi_n$. In probability*

$$\sup_{0 \leqslant t \leqslant 1}\left|n^{-1}\sum_{i=1}^{n}\psi_n'\left(\frac{i}{n+1}\right)\eta_i\zeta_{in}(t)-tn^{1/2}\int_0^1 \psi_n' g\right| \to 0.$$

The assertion remains valid, if $\zeta_{in}(t)$ is replaced by $\bar{\zeta}_{in}(t)$.

Proof. We shall show that for any $0<\mu<1$ in probability

$$\sup_t \left| n^{-1} \sum_{i=n-n^\mu}^{n} \varphi'\left(\frac{i}{n+1}\right) \eta_i \zeta_{in}(t) - tn^{1/2} \int_{1-n^{\mu-1}}^{1} \varphi' g \right| \to 0. \qquad (26)$$

Along the same lines it follows that (26) remains true, if φ is replaced by φ_n. In view of (15) this entails our assertion. First, we show (26) in the case $\mu < (\rho - \frac{1}{2})/(\rho+1)$. In view of (5) $(\mu-1)(\rho-\lambda)<\frac{1}{2}$, therefore from (3) and (4),

$$\int_{1-n^{\mu-1}}^{1} \varphi' g = O\left(\int_0^{n^{\mu-1}} t^{\rho-\lambda-1}\,dt\right) = o(n^{-1/2}).$$

Next suppose $\sum_{i\geqslant n-n^\mu} \varphi'(i/(n+1))\,\eta_i \zeta_{in}(1) \neq 0$. By definition of $\zeta_{in}(t)$ for at least one $i \geqslant n-n^\mu$ we have $G(T_{i+1}T_{n+1}^{-1}) \leqslant G(T_i T_{n+1}^{-1}) + 2n^{-1/2}$. For any $\delta>0$ in view of (25) this event is contained in

$$A_n = \{T_{i+1} - T_i \leqslant n^{1/2+(\mu-1)\rho+\delta} \text{ for at least one } n-n^\mu \leqslant i \leqslant n\},$$

up to an event, which asymptotically vanishes. Since $T_{i+1}-T_i$ are i.i.d. and exponentially distributed, $P(A_n) \leqslant n^{\mu+1/2+(\mu-1)\rho+\delta} = o(1)$, if $\delta>0$ is small enough, since $\mu<(\rho-\frac{1}{2})/(\rho+1)$. Thus $\sum_{i\geqslant n-n^\mu} \varphi'(i/(n+1))\,\eta_i\zeta_{in}(t)=0$ for all $t\in[0,1]$ with probability going to one and (26) follows.

It remains to show that in probability

$$\sup_t \left| n^{-1} \sum_{i=n-n^\mu}^{n-n^\nu} \varphi'\left(\frac{i}{n+1}\right) \eta_i \zeta_{in}(t) - tn^{1/2} \int_{1-n^{\mu-1}}^{1-n^{\nu-1}} \varphi' g \right| \to 0, \qquad (27)$$

where $0<\nu<\mu<1$ and $\nu<(\rho-\frac{1}{2})/(\rho+1)$. To this end let, for fixed $t\in[0,1]$, k be the largest natural number j such that $G(T_j T_{n+1}^{-1}) \leqslant G(T_i T_{n+1}^{-1}) + 2tn^{-1/2}$. In view of Lemma 10 there is a $\xi_{in}(t)$ between ξ_{ikn} and $\xi_{i(k+1)n}$ such that

$$\zeta_{in}(t) = \text{card}\left\{ j \mid i<j\leqslant n,\ T_j \leqslant T_i + 4tn^{1/2} g\left(\frac{i}{n+1}\right)\xi_{in}(t),\ \eta_j=0 \right\}$$

and

$$\sup\{|\xi_{in}(t)-1| : n-n^\mu \leqslant i \leqslant n-n^\nu,\ 0\leqslant t\leqslant 1\} = O_p(n^{\nu-\mu/2+\delta})$$

for all $\delta>0$ and $0<\nu<\mu<1$ such that $\nu>\mu/2$. Now (27) follows from Lemma 9. ∎

LEMMA 12. *Let τ_{in} be random variables such that*

$$\left|\tau_{in} - \frac{i}{n+1}\right| \leqslant n^{-1}(\zeta_{in}(1) + \bar{\zeta}_{in}(1)).$$

Then in probability

$$\sum_{i=1}^{n} \psi_n''(\tau_{in})(\zeta_{in}^2(1) + \bar{\zeta}_{in}^2(1)) = o_p(n^{-2}).$$

Proof. Similar as in the proof of Lemma 11 it is sufficient to show

$$n^{-2} \sum_{i=n-n^{\mu}}^{n} \varphi''(\tau_{in})(\zeta_{in}^2(1) + \bar{\zeta}_{in}^2(1)) \to 0$$

in probability for any $0 < \mu < 1$. If $\mu < (\rho - \frac{1}{2})(\rho + 1)^{-1}$, using arguments of the proof of Lemma 11, $\sum_{i \geqslant n-n^{\mu}} \varphi''(\tau_{in}) \zeta_{in}^2(1) = 0$ with probability going to 1, as $n \to \infty$. Thus we have to show

$$\sum_{i=n-n^{\mu}}^{n-n^{\nu}} \varphi''(\tau_{in}) \zeta_{in}^2(1) = o_p(n^{-2}) \tag{28}$$

with $0 < \nu < \mu < 1$, if only $\mu - \nu$ is small enough.

Now, if $G(T_j T_{n+1}^{-1}) \leqslant G(T_i T_{n+1}^{-1}) + 2n^{-1/2}$, $T_j - T_i \leqslant 2n^{\mu/2}$ in view of (25) and $j - i \leqslant 4n^{\mu/2}$ ($T_j - T_i$ is the sum of $j - i$ i.i.d. random variables with mean 1) up to an asymptotically negligible event. Thus $\zeta_{in}(1) \leqslant 4n^{\mu/2} \leqslant \frac{1}{4}n^{\nu}$ and $\bar{\zeta}_{in}(1) \leqslant \frac{1}{4}n^{\nu}$ with probability going to 1, if $\mu - \nu$ is small enough.

Consequently $|1 - \tau_{in}| \geqslant \frac{1}{2}n^{\nu - 1}$, and in view of Lemma 10, with probability tending to 1,

$$n^{-2} \sum_i{}' \varphi''(\tau_{in}) \zeta_{in}^2(1)$$
$$= O\left(n^{-2-(\nu-1)(2+\lambda)} \sum_i{}' \operatorname{card}\left\{j > i \mid T_j \leqslant T_i + 8n^{1/2} g\left(\frac{i}{n+1}\right)\right\}^2\right).$$

The dash indicates that the sum is taken over all i with $n - n^{\mu} \leqslant i \leqslant n - n^{\nu}$. The mean of the right-hand side is of order n^r with $r = (\mu - 1)(2\rho - 1 - \lambda) + (\mu - \nu)(2 + \lambda)$, $r < 0$, because of (5), if only $\mu - \nu$ is small enough, and (28) follows. ∎

Remark. Ties, concerning the ranks $R_i(t)$, occur, if $|X_i - tn^{-1/2}| = |X_j - tn^{-1/2}|$, i.e., $X_i + X_j = 2tn^{-1/2}$ for some $i \neq j$. With positive probability there are $t \in [0, 1]$ with this property, but a.s. this happens for each t at most for two numbers i, j. Let $R_i = k \geqslant n/2$. If for $i \neq j$ $X_i + X_j = 2tn^{-1/2}$, $C_i(t) \neq 0$. Using the notations of the proof of Lemma 7, $C_i(1) = D_k(1) = \eta_k \zeta_{kn}(1) + \bar{\eta}_k \bar{\zeta}_{kn}(1)$. As was shown in the proof of Lemma 11, $\zeta_{kn}(1) = \bar{\zeta}_{kn}(1) = 0$ for all $k \geqslant n - n^{\mu}$ up to a negligible event, if only $\mu < (\rho - \frac{1}{2})(\rho + 1)^{-1}$. Thus we may assume that $X_i + X_j = 2tn^{-1/2}$ entails $R_i(t) \leqslant n - n^{\mu}$ for any such μ. If now a tie occurs, $R_i(t)$ and $R_j(t)$ are well defined up to an amount of 1, depending on how we relate the ranks to $|X_i - tn^{-1/2}|$ and $|X_j - tn^{-1/2}|$. The corresponding modification is at most of the magnitude

$$2\left|\varphi\left(\frac{R_i(t)}{n+1}\right) - \varphi\left(\frac{R_i(t)\pm 1}{n+1}\right)\right| \leqslant \frac{2}{n+1}\varphi'\left(\frac{R_i(t)+1}{n+1}\right) = o(n^s),$$

where $s = -1-(\mu-1)(1+\lambda)$. If μ is chosen close enough to $(\rho-\frac{1}{2})(\rho+1)^{-1}$, $s<0$, because of (5). Thus different treatments of ties have no consequences.

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